Algebraic D-modules - working group seminar WS13/14

Arbeitsgruppe Algebra & Darstellungstheorie

October 22, 2013

In this seminar we aim to understand the basics of algebraic D-module theory following the book [3] by Countinho. We will mainly work with modules over the Weyl algebra $A_n(K)$ over a field K of characteristic zero. After discussing some general theory of this ring and how its modules can be related to analytic problems, we will develop a theory of dimension for finitely generated modules over $A_n(K)$. One of our main theorems will be Bernstein's inequality:

Theorem 1. *Let M be a finitely generated $A_n(K)$-module with dimension $d(M)$. Then the following inequalities are always satisfied:*

$$n \leq d(M) \leq 2n$$

In the course of this seminar, we will see three proofs of this result.
The modules whose dimension is as small as possible are called holonomic and are of great importance in many areas of mathematics. Any such module is cyclic and of finite length as we will see.
There is also a more geometric way to understand the concept of dimension, namely through the so-called characteristic varieties. A further goal of this seminar is to understand the definition of the characteristic variety $\mathsf{ch}(M)$ of a finitely generated $A_n(K)$-module M and prove the following result:

Theorem 2. $\mathsf{dim}(\mathsf{ch}(M)) = d(M)$

Using symplectic geometry, we will characterise holonomic modules via their characteristic varieties and construct examples of simple $A_n(K)$-modules which are not holonomic.
In the last two talks, which will not include full proofs, we will then see how to generalise the theory of holonomic modules and characteristic varieties, to the sheaves of differential operators over smooth complex varieties and how D-modules were used in the proofs of the Kazhdan-Lusztig conjecture.

1 The Weyl algebra (Lara Bossinger) 5.11.2013

definition as polynomial differential operators, presentation with generators and equations, equivalence of the two in characteristic zero, skew polynomial rings + Hilbert basis theorem, Bernstein filtration and its associated graded ring, simplicity of Weyl algebra in characteristic zero, Stafford's theorem (without proof)
References: chapters 1, 2 and 7 of [3], chapter 1 and 6.6.14 of [7], chapter 1 of [5], sections 1 and 7 of chapter 1 of [2]

2 Modules over the Weyl algebra I - examples (Christian Desczyk) 12.11.2013

no finite dimensional modules in characteristic zero, construct modules via differential equations, Hom = solution space, twisting of modules (in particular: the Fourier transform), Bernstein's functional equation (without proof, this will follow later in the seminar), Bernstein-Sato polynomials, meromorphic continuation of certain integrals
References: chapters 5 and 6 of [3], lecture 1 of [4], chapter 5 of [1], chapter 7 of [6]

3 Modules over the Weyl algebra II - dimension and multiplicity (Deniz Kus) 19.11.2013

good filtrations, Hilbert polynomial, dimension and multiplicity and their independence of the choice of a good filtration, behaviour in short exact sequences, lemma 10.3.1 in [3] or one of its variants
References: chapter 7-9 of [3], chapter 8 of [7] section 4 onwards, sections 2 and 3 of chapter 1 of [2]

4 Modules over the Weyl algebra III - holonomy (Christian Reinecke) 26.11.2013

Bernstein's inequality, definition of holonomic module, cyclicity and finite length of holonomic modules, examples (holonomic modules over the first Weyl algebra, localisation)
References: chapter 10 of [3], section 5 of chapter 12 of [3], section 8 of chapter 1 of [8]

5 Modules over the Weyl algebra IV - homological algebra (Wassilij Gnedin) 10.12.2013

global dimensions of skew polynomial rings, global dimension of the Weyl algebra, proof of Bernstein's inequality via Ext-groups
References: chapter 7 of [7], chapter 2 of [2]

6 Characteristic varieties I - reconciliation (Jacinta Perez Gavilan) 14.1.2014

definition of characteristic variety, behaviour with respect to short exact sequences, dimension of variety agrees with the algebraic dimension of the module
References: section 7 of chapter 1 of [8]

7 Characteristic varieties II - symplectic geometry (Andreas Hochenegger) 14.1.2014

Poisson bracket and involutive varieties, third proof of Bernstein's inequality (admitting Gabber's theorem), non-holonomic irreducible modules
References: chapter 11 of [3]

8 D-modules on algebraic varieties (Lennart Galinat) 21.1.2014

overview of D-modules on smooth complex varieties following chapters 2-5 in [8]: definitions, characteristic variety, Kashiwara's theorem, the example of projective spaces, Beilinson-Bernstein localisation(?)

9 Outlook: Kazhdan-Lusztig conjecture, etc... (Alexander Alldridge) 28.1.2014

References

[1] J. Ayoub, Introduction to Algebraic D-Modules, available at http://www.math.uzh.ch/index.php?file&key1=14027.

[2] J.-E. Björk, Rings of Differential Operators, North-Holland Mathematical Library Vol. **21**, Elsevier.

[3] S.C. Coutinho, A Primer of Algebraic D-modules, LMS Students Texts **33**, Cambridge University Press.

[4] P. Etingof, Lecture Notes for math 18.769, Fall 2013, available on the author's homepage.

[5] K.R. Goodearl & R.B. Warfield, An Introduction to Noncommutative Noetherian Rings (Second Edition), LMS Student Texts **61**, Cambridge University Press.

[6] V. Guillemin & S. Sternberg, Geometric Asymptotics, Mathematical Surveys and Monographs **14**, American Mathematical Society.

[7] J.C. McConnell & J.C. Robson, Noncommutative Noetherian Rings, Graduate Studies in Mathematics **30**, American Mathematical Society.

[8] D. Miličić, Lectures on Algebraic Theory of D-Modules, available on the author's homepage.